6.1 DISCRETE TIME-FREQUENCY DISTRIBUTIONS [0]

For the purposes of digital storage and processing, any real-life signal that is not discrete-time and time-limited must be made so by sampling and windowing. Moreover, if we wish to evaluate a continuous time-frequency distribution (TFD) numerically in a finite number of operations, we must be content with computing a sampled form of the TFD from a finite number of samples of the signal. For such reasons, we need to define discrete, time-limited equivalents of continuous TFDs. This article derives discrete forms of the Wigner-Ville distribution (WVD), the windowed WVD and the general quadratic TFD, and gives thirteen examples of discrete-time kernels. Thus it extends the material present in Chapters 2 and 3.

If the signal $z(t)$ is ideally sampled at times $t = n/f_\mathrm{s}$, where n is an integer and f_s is the sampling rate, it becomes

$$z(t) \sum_{n=-\infty}^{\infty} \delta\big(t - \tfrac{n}{f_\mathrm{s}}\big) = \sum_{n=-\infty}^{\infty} z\big(\tfrac{n}{f_\mathrm{s}}\big)\, \delta\big(t - \tfrac{n}{f_\mathrm{s}}\big) \tag{6.1.1}$$

where $\delta(\ldots)$ denotes the unit impulse function. We shall use a wide caret $(\widehat{\ })$ to denote a TFD that has been modified by sampling. In the lag (τ) domain, we shall consider only *ideal* sampling at $\tau = m/f_\mathrm{s}$, where m is an integer. This in turn will draw attention to the discrete time values $t = n/f_\mathrm{s}$, where n is an integer; but it will not be necessary to specify ideal sampling in the time domain.

6.1.1 The Discrete Wigner-Ville Distribution (DWVD)

The WVD of a continuous-time signal $z(t)$ has the form

$$W_z(t,f) = \underset{\tau\to f}{\mathcal{F}}\big\{z(t+\tfrac{\tau}{2})\, z^*(t-\tfrac{\tau}{2})\big\} = \int_{-\infty}^{\infty} z(t+\tfrac{\tau}{2})\, z^*(t-\tfrac{\tau}{2})\, e^{-j2\pi f\tau} d\tau. \tag{6.1.2}$$

Changing the variable of integration to $\theta = \tau/2$, we obtain the alternative definition

$$W_z(t,f) = 2\int_{-\infty}^{\infty} z(t+\theta)\, z^*(t-\theta)\, e^{-j4\pi f\theta} d\theta \tag{6.1.3}$$

which we shall find more convenient for the purpose of conversion to discrete form. The product $z(t{+}\theta)\, z^*(t{-}\theta)$ is the instantaneous autocorrelation function (IAF) in terms of t and θ. Let us consider this IAF as a function of θ and suppose that it is sampled.

Theorem 6.1.1: If $W_z(t,f)$ is modified by ideally sampling the IAF in θ at $\theta = \tau/2 = m/f_\mathrm{s}$, where m is an integer and f_s is the sampling rate, and if the

[0] Authors: **Boualem Boashash** and **Gavin R. Putland**, Signal Processing Research Centre, Queensland University of Technology, GPO Box 2434, Brisbane, Q 4001, Australia (b.boashash@qut.edu.au, g.putland@qut.edu.au). Reviewers: G. Matz and LJ. Stanković.

modified TFD is denoted by $\widehat{W}_z(t,f)$, then

$$\widehat{W}_z\big(\tfrac{n}{f_s}, \tfrac{kf_s}{2N}\big) = 2 \sum_{|m|<N/2} z\big(\tfrac{n+m}{f_s}\big)\, z^*\big(\tfrac{n-m}{f_s}\big)\, e^{-j2\pi km/N} \tag{6.1.4}$$

where N is the duration (in samples) of $z(n/f_s)$; and the time support for $\widehat{W}_z$ is the same as for $z(n/f_s)$.

Proof/explanation: After sampling, the integrand in Eq. (6.1.3) becomes

$$z(t+\theta)\, z^*(t-\theta)\, e^{-j4\pi f\theta} \sum_{m=-\infty}^{\infty} \delta\big(\theta - \tfrac{m}{f_s}\big) \tag{6.1.5}$$

and the WVD becomes

$$\widehat{W}_z(t,f) = 2 \sum_{m=-\infty}^{\infty} z\big(t+\tfrac{m}{f_s}\big)\, z^*\big(t-\tfrac{m}{f_s}\big)\, e^{-j4\pi fm/f_s}. \tag{6.1.6}$$

Now let t be conveniently restricted to $t = n/f_s$ where n is an integer (this represents sampling of the signal in time, but the sampling is not ideal). Then, if $z(t)$ has finite duration, the time support of $z(n/f_s)$ can be written in the form $|n - n_0| < N/2$, where n_0 is real. Hence the time support of $\widehat{W}_z$ is

$$|n \pm m - n_0| < N/2 \tag{6.1.7}$$

where both signs must be satisfied. Because the sign can be chosen so as to increase the magnitude, we must have $|n - n_0| + |m| < N/2$, hence $|n - n_0| < N/2$ and $|m| < N/2$. So the time support for $\widehat{W}_z$ is the same as for $z(n/f_s)$, while the summation in Eq. (6.1.6) is restricted to $|m| < N/2$, giving a maximum of N terms.[1] The sampling in τ makes $\widehat{W}_z(t,f)$ periodic in f with period $f_s/2$, while the time-limiting gives a frequency resolution of N bins per period. So it is convenient to let

$$f = \frac{kf_s}{2N} \tag{6.1.8}$$

where k is an integer. With these restrictions, Eq. (6.1.6) reduces to Eq. (6.1.4). ■

With a change of notation, Eq. (6.1.4) becomes

$$W_z[n,k] = 2 \sum_{|m|<N/2} z[n+m]\, z^*[n-m]\, e^{-j2\pi km/N}. \tag{6.1.9}$$

This $W_z[n,k]$ is called the **discrete WVD** or **DWVD**. If $z[n]$ has a duration not exceeding N samples, the DWVD is represented as an $N \times N$ real matrix. If the summand is extended periodically in m with period N (i.e. extended periodically in τ with period $2N/f_s$), we obtain

$$W_z[n,k] = 2 \underset{m\to k}{\mathrm{DFT}} \{z[n+m]\, z^*[n-m]\} \ ; \quad m \in \langle N\rangle \tag{6.1.10}$$

where $\langle N\rangle$ means any set of N consecutive integers.[2]

[1] N terms for odd N; $N-1$ terms for even N.

[2] For even N, the periodic extension is padded with a zero term.

It remains to find the minimum value of f_s that avoids aliasing in the frequency and Doppler domains. As usual, we define the instantaneous autocorrelation function as $K_z(t,\tau) = z(t+\frac{\tau}{2})\, z^*(t-\frac{\tau}{2})$, the spectrum as $Z(f) = \underset{t\to f}{\mathcal{F}}\{z(t)\}$, the WVD as $W_z(t,f) = \underset{\tau\to f}{\mathcal{F}}\{K_z(t,\tau)\}$, and the ambiguity function as $A_z(\nu,\tau) = \underset{t\to\nu}{\mathcal{F}}\{K_z(t,\tau)\}$. Using these notations and the familiar properties of the FT, it can be shown that

$$W_z(t,f) = \left[2Z(2f)\, e^{j4\pi f t}\right] \underset{f}{*} \left[2Z^*(2f)\, e^{-j4\pi f t}\right] \tag{6.1.11}$$

$$A_z(\nu,\tau) = \left[Z(\nu)\, e^{j\pi\nu\tau}\right] \underset{\nu}{*} \left[Z^*(-\nu)\, e^{-j\pi\nu\tau}\right]. \tag{6.1.12}$$

If $Z(f)$ is zero outside the band $|f| < B/2$, then the spectrum on the right of Eq. (6.1.11), i.e. the WVD, is zero outside the band $|f| < B/2$, and the spectrum on the right of Eq. (6.1.12) is zero outside the band $|\nu| < B$. Aliasing will be avoided if the sampling rate in τ (namely $f_\mathrm{s}/2$, since f_s is the sampling rate in θ) is at least B and the sampling rate in t (namely f_s) is at least $2B$; that is, aliasing will be avoided if $f_\mathrm{s} \leq 2B$. The minimum sampling rate in τ makes the WVD periodic in f with period B.

The sampling rate may be reduced in the case of an analytic signal. If $z(t)$ is the analytic associate of the real signal $s(t)$, whose spectrum $S(f)$ is zero outside the band $|f| < B/2$, then $Z(f)$ is zero outside the band $0 \leq f < B/2$, so that the spectrum on the right of Eq. (6.1.11), i.e. the WVD, is zero outside the band $0 \leq f < B/2$, and the spectrum on the right of Eq. (6.1.12) is zero outside the band $|\nu| < B/2$. Aliasing will be avoided if the sampling rate in τ is at least $B/2$ and the sampling rate in t is at least B; that is, aliasing will be avoided if $f_\mathrm{s} \leq B$. The minimum sampling rate in τ makes the WVD periodic in f with period $B/2$.

Now consider $W_z[n,k]$ as a matrix with k as the column index. If the sampling rate in t is B, there is no zero-padding in the frequency domain for an analytic signal, and all the columns of $W_z[n,k]$ are needed for the positive frequencies. If the sampling rate in t is $2B$, only half the columns of $W_z[n,k]$ are needed to represent the positive frequencies. The negative-frequency elements may be assumed to be zero if $z[n]$ is analytic, provided of course that the assumption of analyticity is not invalidated by short-segment effects, such as windowing of the signal in the n domain and/or windowing of the IAF in the m domain. The latter kind of windowing is discussed under the next heading.

6.1.2 The Windowed DWVD

Nonlinear IF laws and multiple signal components give rise to artifacts (interference terms, cross-terms) in the WVD [see Article 4.2]. The effect of nonlinear IF laws can be reduced, with a concomitant loss of frequency resolution, by windowing the IAF in the lag direction (i.e. in τ or θ) before taking the FT. For continuous time, if the window is

$$g(\tau) = \bar{g}(\tfrac{\tau}{2}) = \bar{g}(\theta)\ , \tag{6.1.13}$$

then the resulting TFD, denoted by $W_z^g(t,f)$, is

$$W_z^g(t,f) = \int_{-\infty}^{\infty} g(\tau)\, z(t+\tfrac{\tau}{2})\, z^*(t-\tfrac{\tau}{2})\, e^{-j2\pi f\tau} d\tau \tag{6.1.14}$$

$$= 2\int_{-\infty}^{\infty} \bar{g}(\theta)\, z(t+\theta)\, z^*(t-\theta)\, e^{-j4\pi f\theta} d\theta \tag{6.1.15}$$

and is called the **windowed WVD** or **pseudo-WVD**. The effects of sampling and time-limiting on the windowed WVD are described by the following theorem.

Theorem 6.1.2: If $W_z^g(t,f)$ is modified by ideally sampling $g(\tau)$ in θ at $\theta = \tau/2 = m/f_s$, where m is an integer and f_s is the sampling rate, and if $g(\tau)$ is time-limited so that

$$g(\tau) = 0 \quad \text{for} \quad |\theta| = |\tfrac{\tau}{2}| \geq \tfrac{M}{2f_s} \tag{6.1.16}$$

where M is a positive integer, and if the modified TFD is denoted by $\widehat{W}_z^g(t,f)$, then

$$\widehat{W}_z^g\big(\tfrac{n}{f_s}, \tfrac{kf_s}{2M}\big) = 2 \sum_{|m|<M/2} \bar{g}\big(\tfrac{m}{f_s}\big)\, z\big(\tfrac{n+m}{f_s}\big)\, z^*\big(\tfrac{n-m}{f_s}\big)\, e^{-j2\pi km/M} \tag{6.1.17}$$

and the time support for $\widehat{W}_z^g$ is the same as for $z(n/f_s)$.

Proof: Apart from the limits on τ, θ and m, which lead to the substitution $f = \frac{kf_s}{2M}$, the explanation is similar to that of Theorem 6.1.1. ■

With a change of notation, Eq. (6.1.17) becomes

$$W_z^g[n,k] = 2 \sum_{|m|<M/2} g[m]\, z[n+m]\, z^*[n-m]\, e^{-j2\pi km/M}. \tag{6.1.18}$$

This $W_z^g[n,k]$ is called the **windowed DVWD** or **pseudo-DWVD**. If $z[n]$ has a duration not exceeding N samples, the windowed DWVD is represented as an $N \times M$ matrix. If the summand is extended periodically in m with period M (i.e. extended periodically in τ with period $2M/f_s$), we obtain

$$W_z^g[n,k] = 2 \mathop{\mathrm{DFT}}_{m\to k} \{g[m]\, z[n+m]\, z^*[n-m]\} \;;\quad m \in \langle M \rangle\,. \tag{6.1.19}$$

6.1.3 The Discrete Quadratic TFD

In terms of continuous variables, the general quadratic TFD is

$$\rho_z(t,f) = \mathop{\mathcal{F}}_{\tau\to f}\big\{G(t,\tau) \mathop{*}_{t} [z(t+\tfrac{\tau}{2})\, z^*(t-\tfrac{\tau}{2})]\big\} \tag{6.1.20}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u,\tau)\, z(t-u+\tfrac{\tau}{2})\, z^*(t-u-\tfrac{\tau}{2})\, du\, e^{-j2\pi f\tau} d\tau \tag{6.1.21}$$

where $G(t,\tau)$ is the time-lag kernel. If we define

$$\bar{G}(t,\theta) = G(t,\tau) \tag{6.1.22}$$

where $\theta = \tau/2$, we obtain the alternative definition

$$\rho_z(t,f) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{G}(u,\theta)\, z(t-u+\theta)\, z^*(t-u-\theta)\, e^{-j4\pi f\theta} du\, d\theta. \tag{6.1.23}$$

Theorem 6.1.3: If $\rho_z(t,f)$ is modified by ideally sampling $G(u,\tau)$ in θ at $\theta = \tau/2 = m/f_{\mathrm{s}}$ and in u at $u = p/f_{\mathrm{s}}$, where m and p are integers and f_{s} is the sampling rate, and if $G(u,\tau)$ is time-limited so that

$$G(u,\tau) = 0 \quad \text{for} \quad |\theta| = |\tfrac{\tau}{2}| \geq \tfrac{M}{2f_{\mathrm{s}}} \text{ or } |u| \geq \tfrac{P}{2f_{\mathrm{s}}} \tag{6.1.24}$$

where M and P are positive integers, and if the modified TFD is denoted by $\widehat{\rho}_z(t,f)$, then

$$\widehat{\rho}_z\big(\tfrac{n}{f_{\mathrm{s}}}, \tfrac{kf_{\mathrm{s}}}{2M}\big) = 2 \sum_{|m|<\frac{M}{2}} \sum_{|p|<\frac{P}{2}} \bar{G}\big(\tfrac{p}{f_{\mathrm{s}}}, \tfrac{m}{f_{\mathrm{s}}}\big)\, z\big(\tfrac{n-p+m}{f_{\mathrm{s}}}\big)\, z^*\big(\tfrac{n-p-m}{f_{\mathrm{s}}}\big)\, e^{-j2\pi km/M}. \tag{6.1.25}$$

Proof/explanation: The sampled version of $G(u,\theta)$ is

$$\begin{aligned}
\widehat{G}(u,\theta) &= G(u,\theta) \sum_{p=-\infty}^{\infty} \delta\big(u-\tfrac{p}{f_{\mathrm{s}}}\big) \sum_{m=-\infty}^{\infty} \delta\big(\theta-\tfrac{m}{f_{\mathrm{s}}}\big) &(6.1.26)\\
&= G(u,\theta) \sum_{p=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta_2\big(u-\tfrac{p}{f_{\mathrm{s}}}\,,\, \theta-\tfrac{m}{f_{\mathrm{s}}}\big) &(6.1.27)
\end{aligned}$$

where $\delta_2(u,\theta)$ is the two-dimensional unit impulse function. When $G(u,\theta)$ is replaced by $\widehat{G}(u,\theta)$, Eq. (6.1.23) becomes

$$\widehat{\rho}_z(t,f) = 2 \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \bar{G}\big(\tfrac{p}{f_{\mathrm{s}}}, \tfrac{m}{f_{\mathrm{s}}}\big)\, z\big(t-\tfrac{p}{f_{\mathrm{s}}}+\tfrac{m}{f_{\mathrm{s}}}\big)\, z^*\big(t-\tfrac{p}{f_{\mathrm{s}}}-\tfrac{m}{f_{\mathrm{s}}}\big)\, e^{-j4\pi fm/f_{\mathrm{s}}} \tag{6.1.28}$$

The time-limiting of $G(u,\tau)$ restricts the summation to $|m| < M/2$ and $|p| < P/2$. The sampling in τ makes $\rho_z(t,f)$ periodic in f with period $f_{\mathrm{s}}/2$, while the time-limiting in τ (or m) gives a frequency resolution of M bins per period. So it is convenient to let

$$f = \frac{kf_{\mathrm{s}}}{2M} \tag{6.1.29}$$

where k is an integer. With these restrictions, and with $t = n/f_{\mathrm{s}}$, Eq. (6.1.28) reduces to Eq. (6.1.25). ■

With a change of notation, Eq. (6.1.25) becomes

$$\begin{aligned}
\rho_z[n,k] &= 2 \sum_{|m|<\frac{M}{2}} \sum_{|p|<\frac{P}{2}} G[p,m]\, z[n-p+m]\, z^*[n-p-m]\, e^{-j2\pi km/M} &(6.1.30)\\
&= 2 \sum_{|m|<\frac{M}{2}} G[n,m] \underset{n}{*} (z[n+m]\, z^*[n-m])\, e^{-j2\pi km/M}. &(6.1.31)
\end{aligned}$$

This $\rho_z[n,k]$ is the generalized **discrete quadratic TFD**.

If the summand is extended in m with period M, Eq. (6.1.31) becomes

$$\rho_z[n,k] = 2 \underset{m\to k}{\mathrm{DFT}} \left\{ G[n,m] \underset{n}{*} (z[n+m]\, z^*[n-m]) \right\} ;\ m \in \langle M \rangle . \tag{6.1.32}$$

Because the time (n) support for $z[n+m]\, z^*[n-m]$ is the same as for $z[n]$, the time support for $\rho_z[n,k]$ is the same as for $G[n,m] \underset{n}{*} z[n]$. If the latter has a duration not exceeding N samples, then the non-zero elements of $\rho_z[n,k]$ may represented as an $N \times M$ matrix and the linear convolution in Eqs. (6.1.31) and (6.1.32) may be interpreted as modulo-N. If $G[n,m]$ is real and even in m, then the argument of the DFT is Hermitian in m, so that $\rho_z[n,k]$ is real.

Eq. (6.1.32) shows that the implementation of the discrete quadratic TFD involves construction of the discrete IAF, followed by convolution in n with the time-lag kernel, followed by discrete Fourier transformation. These steps may be simplified by taking advantage of symmetries, as explained in [1, 2].

It remains to determine the effect of $G(t,\tau)$ on the required sampling rate. If $\underset{\tau\to f}{\mathcal{F}}\{G(t,\tau)\} = \gamma(t,f)$ and $\underset{t\to\nu}{\mathcal{F}}\{G(t,\tau)\} = g(\nu,\tau)$, we have the familiar results

$$\underset{\tau\to f}{\mathcal{F}}\left\{ g(t,\tau) \underset{t}{*} K_z(t,\tau) \right\} \;=\; \gamma(t,f) ** W_z(t,f) \tag{6.1.33}$$

$$\underset{t\to\nu}{\mathcal{F}}\left\{ g(t,\tau) \underset{t}{*} K_z(t,\tau) \right\} \;=\; g(\nu,\tau)\, A_z(\nu,\tau). \tag{6.1.34}$$

Comparing the above with Eqs. (6.1.11) and (6.1.12), we see that there is an additional spreading of the spectrum in f but not in ν. If $\gamma(t,f)$ is zero outside the band $|f| < B_G/2$, then the total bandwidth of the WVD is increased by B_G, so that the required sampling rate is increased by B_G in τ, or $2B_G$ in θ, and the sampling rate in t must be increased to match.

6.1.3.1 Special Cases

If $G[n,m] = \delta[n]\, g[m]$, then Eq. (6.1.32) reduces to Eq. (6.1.19), so that the discrete quadratic TFD reduces to the windowed DWVD. If, in addition, $g[m] = 1$ (that is, if $G[n,m] = \delta[n]$), then Eqs. (6.1.19) and (6.1.32) reduce to Eq. (6.1.10), so that the windowed DWVD and the discrete quadratic TFD reduce to the DWVD.

Two of the three theorems above concern the sampling of a window or kernel function. Theorem 6.1.1 seems to be an exception in that the entire IAF is sampled (which is possible only in theory, as one cannot compute a continuous IAF in practice). However, because the WVD may be considered as a windowed WVD with $g(\tau) = 1$, Theorem 6.1.1 can be restated in terms of sampling the lag window, like Theorem 6.1.2.

6.1.3.2 Doppler-Frequency Form

The Doppler-frequency form of the general quadratic TFD is

$$\rho_z(t,f) = \mathop{\mathcal{F}^{-1}}_{t\leftarrow\nu}\left\{\mathcal{G}(\nu,f) \mathop{*}_{f} [Z(f+\tfrac{\nu}{2})\, Z^*(f-\tfrac{\nu}{2})]\right\} \tag{6.1.35}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{G}(\eta,\nu)\, Z(f-\eta+\tfrac{\nu}{2})\, Z^*(f-\eta-\tfrac{\nu}{2})\, d\eta\, e^{j2\pi\nu t} d\nu \tag{6.1.36}$$

where $\mathcal{G}(\nu,f)$ is the Doppler-frequency kernel. If we define

$$\bar{\mathcal{G}}(\xi,f) = \mathcal{G}(\nu,f) \tag{6.1.37}$$

where $\xi = \nu/2$, we obtain the alternative definition

$$\rho_z(t,f) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\mathcal{G}}(\xi,\eta)\, Z(f-\eta+\xi)\, Z^*(f-\eta-\xi)\, e^{j4\pi\xi t} d\eta\, d\xi. \tag{6.1.38}$$

As the time-lag definition of the quadratic TFD leads to Theorem 6.1.3, so the Doppler-frequency definition leads to the following result.

Theorem 6.1.4: If $\rho_z(t,f)$ is modified by ideally sampling $\mathcal{G}(\eta,\nu)$ in ξ at $\xi = \nu/2 = \frac{lf_s}{2N}$ and in η at $\eta = \frac{qf_s}{2N}$, where l and q are integers and N is a positive integer and f_s is a positive constant, and if $\mathcal{G}(\eta,\nu)$ is band-limited so that

$$\mathcal{G}(\eta,\nu) = 0 \quad \text{for} \quad |\xi| = |\tfrac{\nu}{2}| \geq \tfrac{Lf_s}{4N} \text{ or } |\eta| \geq \tfrac{Qf_s}{4N} \tag{6.1.39}$$

where L and Q are positive integers, and if the modified TFD is denoted by $\widehat{\rho}_z(t,f)$, then

$$\widehat{\rho}_z\left(\tfrac{n}{f_s}, \tfrac{kf_s}{2N}\right) = 2\sum_{|l|<\frac{L}{2}}\ \sum_{|q|<\frac{Q}{2}} \bar{\mathcal{G}}\left(\tfrac{lf_s}{2N}, \tfrac{qf_s}{2N}\right) Z\left(\tfrac{[k-q+l]f_s}{2N}\right) Z^*\left(\tfrac{[k-q-l]f_s}{2N}\right) e^{j2\pi ln/N}. \tag{6.1.40}$$

Proof: Parallel to the proof of Theorem 6.1.3 and the subsequent discussion. ∎

With a change of notation, Eq. (6.1.40) becomes

$$\rho_z[n,k] = 2\sum_{|l|<\frac{L}{2}}\ \sum_{|q|<\frac{Q}{2}} \mathcal{G}[l,q]\, Z[k-q+l]\, Z^*[k-q-l]\, e^{j2\pi ln/N} \tag{6.1.41}$$

$$= 2\sum_{|l|<\frac{L}{2}} \mathcal{G}[l,k] \mathop{*}_{k} (Z[k+l]\, Z^*[k-l])\, e^{j2\pi ln/N}. \tag{6.1.42}$$

If the summand is extended in l with period L, Eq. (6.1.42) becomes

$$\rho_z[n,k] = 2 \mathop{\mathrm{IDFT}}_{n\leftarrow l}\left\{\mathcal{G}[l,k] \mathop{*}_{k} (Z[k+l]\, Z^*[k-l])\right\};\ l\in\langle L\rangle\,. \tag{6.1.43}$$

The frequency support for $\rho_z[n,k]$ is the same as for $G[l,k] \mathop{*}_{k} Z[k]$. If the latter support has a width not exceeding K frequency samples, then the non-zero elements of $\rho_z[n,k]$ may be represented as an $L\times K$ matrix and the linear convolution in Eqs. (6.1.42) and (6.1.43) may be interpreted as modulo-K. While the dimensions of the TFD matrix seem to differ from those in Theorem 6.1.3, the dimensions are upper bounds and may be matched, if desired, by zero-padding.

Table 6.1.1: Kernel requirements for selected properties ("Prop.") of quadratic TFDs, in time-lag and Doppler-frequency domains, for general continuous and discrete TFDs, and discrete TFDs with Doppler-independent (DI) and lag-independent (LI) kernels. Note that $\theta = \tau/2$ and $\xi = \nu/2$.

	KERNEL CONSTRAINTS			
Prop.	Continuous	Discrete	Discrete DI $G[n,m]=\delta[n]g_2[m]$ $\mathcal{G}[l,k] = G_2[k]$	Discrete LI $G[n,m] = g_1[n]$ $\mathcal{G}[l,k]=G_1[l]\delta[k]$
RE:	$G(t,\tau)$ $= G^*(t,-\tau)$.	$G[n,m]$ $= G^*[n,-m]$.	$g_2[m] = g_2^*[-m]$	$g_1[n]$ is real.
TM:	$G(t,0) = \delta(t)$	$G[n,0] = \delta[n]$	$g_2[0] = 1$	WVD only
FM:	$\mathcal{G}(0,f) = \delta(f)$	$\mathcal{G}[0,k] = \delta[k]$	WVD only	$G_1[0] = 1$
IF:	$G(t,0) = \delta(t)$; $\int f\mathcal{G}(\nu,f)\,df = 0$.	$G[n,0] = \delta[n]$; $\sum_k k\mathcal{G}[l,k] = 0$.	$g_2[0] = 1$; $\sum_k kG_2[k] = 0$.	WVD only
TD:	$\mathcal{G}(0,f) = \delta(f)$; $\int tG(t,\tau)\,dt = 0$.	$\mathcal{G}[0,k] = \delta[k]$; $\sum_n nG[n,m] = 0$.	WVD only	$G_1[0] = 1$; $\sum_n ng_1[n] = 0$.
TS:	$G(t,\tau) = 0$ if $\lvert\theta\rvert < \lvert t\rvert$.	$G[n,m] = 0$ if $\lvert m\rvert < \lvert n\rvert$.	Always	WVD only
FS:	$\mathcal{G}(\nu,f) = 0$ if $\lvert\xi\rvert < \lvert f\rvert$.	$\mathcal{G}[l,k] = 0$ if $\lvert l\rvert < \lvert k\rvert$.	WVD only	Always

6.1.4 Desirable Properties; Kernel Constraints

The desirable properties of continuous TFDs (defined in Section 3.1.1, p. 60 ff) are easily redefined for discrete TFDs. Some important examples are given below.

Realness (RE) says simply that the TFD is real.

The **marginal conditions**, which may be considered optional for signal-processing purposes, are the **time marginal (TM)**

$$\textstyle\sum_k \rho_z[n,k] = |z[n]|^2 \tag{6.1.44}$$

and the **frequency marginal (FM)**

$$\textstyle\sum_n \rho_z[n,k] = |Z[k]|^2. \tag{6.1.45}$$

The **IF property** says that the periodic first moment of the TFD w.r.t. frequency is the instantaneous frequency. Its dual, which seems to be regarded as less important, is the **time delay property (TD)**, and says that the periodic first moment of the TFD w.r.t. time is the time delay.

The **time support (TS)** property says that if $z[n] = 0$ everywhere except $n_1 \le n \le n_2$, then $\rho_z[n,k] = 0$ everywhere except $n_1 \le n \le n_2$. Similarly, the

Table 6.1.2: Kernels of selected TFDs in time-lag and Doppler-frequency domains. For the spectrogram and windowed Levin (w-Levin) distributions, the window $w[n]$ is assumed to be real and even. $W[k]$ denotes the sampled spectrum of the window $w(\tau)$. In the PROPERTY column, an exclamation (!) means that the property is always satisfied, while an asterisk (*) means that the property is satisfied subject to non-degenerate constraints on the window and/or parameter.

	KERNEL		PROPERTY						
Distribution	$G[n,m]$	$\mathcal{G}[l,k]$	RE	TM	FM	IF	TD	TS	FS
WVD	$\delta[n]$	$\delta[k]$	!	!	!	!	!	!	!
Levin	$\frac{1}{2}\delta[n{+}m]$ $+\frac{1}{2}\delta[n{-}m]$	$\frac{1}{2}\delta[k{+}l]$ $+\frac{1}{2}\delta[k{-}l]$	!	!	!	!	!	!	!
Born-Jordan	$\left[\frac{1}{4\lvert\alpha m\rvert}\operatorname{rect}\left(\frac{n}{4\alpha m}\right)\right]$ $**\left[\operatorname{sinc} n \operatorname{sinc} m\right]$	$\left[\frac{1}{4\lvert\alpha l\rvert}\operatorname{rect}\left(\frac{k}{4\alpha l}\right)\right]$ $**\left[\operatorname{sinc} k \operatorname{sinc} l\right]$	!	!	!	!	!	*	*
Modified B	$\frac{\cosh^{-2\beta} n}{\sum_n \cosh^{-2\beta} n}$		!		!		!		!
windowed WVD	$\delta[n]\, w[m]$	$W[k]$	*	*		*		!	
w-Levin	$\frac{1}{2}w[m]\,\delta[n{+}m]$ $+\frac{1}{2}w[m]\,\delta[n{-}m]$	$\frac{1}{2}W[k{+}l]$ $+\frac{1}{2}W[k{-}l]$	!	*		*		!	
Zhao-Atlas-Marks	$\left[w[m]\operatorname{rect}\left(\frac{an}{4m}\right)\right]$ $**\left[\operatorname{sinc} n \operatorname{sinc} m\right]$		*	*		*		*	
Rihaczek	$\delta[n-m]$	$\delta[k+l]$		!	!			!	!
w-Rihaczek	$w^*[-m]\,\delta[n-m]$	$W^*[k+l]$		*				!	
Page	$\delta[n-\lvert m\rvert]$	$\frac{1}{2}\delta[k{+}l]+\frac{1}{2}\delta[k{-}l]$ $+\frac{jl}{\pi(k^2-l^2)} * \operatorname{sinc} l$	!	!	!	!		!	
Choi-Williams	$\frac{\sqrt{\pi\sigma}}{2\lvert m\rvert}\exp\left(\frac{-\pi^2\sigma n^2}{4m^2}\right)$ $**\left[\operatorname{sinc} n \operatorname{sinc} m\right]$	$\frac{\sqrt{\pi\sigma}}{2\lvert l\rvert}\exp\left(\frac{-\pi^2\sigma k^2}{4l^2}\right)$ $**\left[\operatorname{sinc} k \operatorname{sinc} l\right]$	!	!	!	!	!		
B	$\left(\frac{\lvert 2m\rvert}{\cosh^2 n}\right)^{\beta} * \operatorname{sinc} m$		!						
spectrogram	$w[n{+}m]\, w[n{-}m]$	$W[k{+}l]\, W[k{-}l]$	!						

frequency support (FS) property says that if $Z[k]=0$ everywhere except $k_1 \le k \le k_2$, then $\rho_z[n,k]=0$ everywhere except $k_1 \le k \le k_2$.

The class of TFDs satisfying realness, the time and frequency marginals, the IF property and the time and frequency support properties is called class $\mathcal{P}$.

The properties listed above are selected from Table 3.3.1 on p. 75. For each property, Table 3.3.1 gives necessary and sufficient conditions on the kernel of the general continuous quadratic TFD. To obtain the corresponding conditions for discrete TFDs, we first express the conditions entirely in the time-lag and Doppler-frequency domains, using the scaled variables θ and ξ where convenient, obtaining the "Continuous" column of Table 6.1.1. Then we sample the kernels as specified

in Theorems 6.1.3 and 6.1.4, obtaining the "Discrete" column of Table 6.1.1. The remaining columns are obtained by specialization.

The sampling of the time-lag kernel will be free of aliasing if the kernel is first band-limited in t and θ to $\pm f_s/2$. This causes $\delta(t)$ to be discretized as $\delta[n]$. Similarly, the sampling of the Doppler-frequency kernel will be free of aliasing if the kernel is first time-limited in f and ξ to $\pm N/f_s$. This causes $\delta(f)$ to be discretized as $\delta[k]$.

6.1.5 Examples

By sampling the kernels of common continuous quadratic TFDs [see Table 3.3.2 on p. 76], we obtain the new Table 6.1.2, which lists the kernels for the discrete versions of those TFDs. The convolutions with sinc functions are performed *before* restricting the variable to integer values; this requires oversampling and is computationally inefficient. The convolutions in n and m arise from the band-limiting of the time-lag kernel prior to sampling. This band-limiting of the kernel does *not* affect the result of convolving the kernel with the IAF, because the IAF is assumed to be similarly band-limited. Similarly, the convolutions in k and l arise from the time-limiting of the Doppler-Frequency kernel prior to sampling. Where no convolution appears in the kernel, either a sinc function has been converted to a discrete delta function by the sampling, or a window function is assumed to provide sufficient filtering.

6.1.6 Summary and Conclusions

Ideal sampling of window or kernel functions leads to straightforward definitions of discrete-time forms of the WVD, the windowed WVD and other quadratic TFDs. Use of the analytic signal minimizes the required sampling rate.

Further theoretical details may be found in [3–5]. Some practical computational issues will be examined in Article 6.5.

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